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Continuity of the Strong Unicity Constant on $C(X)$ for Changing X

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1. INTRODUCTION

Let $C(X)$ denote the set of real valued continuous functions on the compact metric space X and let $M \subseteq C(X)$ denote a Haar subspace of dimension K. For any compact metric subspace Y of X, let $\| \cdot \|_Y$ denote the uniform norm on Y and let $B_Y(f)$ denote the best uniform approximation to f on Y from M. Then the well-known strong unicity theorem, introduced for uniform approximation in [12], says that for any subset Y of X there exists a constant $\gamma = \gamma(Y, M, f)$ such that for all m in M,

$$\| f - m \|_Y \geqslant \| f - B_Y(f) \|_Y + \gamma \| B_Y(f) - m \|_Y . \tag{1.1}$$

As usual, we take γ to be the largest number for which (1.1) is valid for all $m \in M$.

Several recent papers have studied this $\gamma = \gamma(Y, M, f)$ (see references). Methods of computing γ were given in [2] and [4]. In [1], the continuity properties of γ as a function of f were studied and in [2] a uniform strong unicity constant was found for all f (assuming X was finite). The behavior of γ as a function of M has been considered in [8], [13], and [14]. More precisely, $\lim_{n \to \infty} \gamma(Y, M_n, f)$ was studied where M_n was a Haar space of dimension n. Strong unicity on nearby sets was considered in [5], and in [7] the behavior of γ was studied when X was an interval whose length decreases.

The present paper is concerned with the properties of $\gamma(X, M, f)$ as a

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function of X. For any two subsets A and B of X, the "density" of A in B (cf. [3]) is,

$$d(A, B) = \sup_{y \in B} \inf_{x \in A} d(x, y).$$

We show that under suitable circumstances (see Theorem 1) $\gamma(Y, M, f)$ converges to $\gamma(X, M, f)$ as $d(Y, X)$ converges to zero. The crucial consideration concerns the behavior or number (see Theorem 2) of the extreme points for the best approximations.

The set of extreme points of $f - B_Y(f)$ on Y is:

$$E_Y(f) = \{x \in Y: |f(x) - B_Y(f)(x)| = \| f - B_Y(f)\|_Y\}.$$

When $Y = X$, $E_X(f)$ is denoted by $E(f)$. Finally, let π_K denote the set of polynomials of degree $\leqslant K$.

2. Computing γ and a Counter Example

Two methods will be used to calculate $\gamma(X, f)$. The first [2] is

$$\gamma(X, f) = \inf\{\max_{x \in E(f)} \operatorname{sgn}(f(x) - B_X(f)(x))\, m(x): \| m \|_X = 1\}. \tag{2.1}$$

The second is an observation of M. Henry and J. Roulier [8] based on work of A. Cline [4] in the special case when $E(f)$ has $K + 1$ points. In this case let $\{x_k\}_{k=0}^K$ be the points in $E(f)$. Then for each $i = 0,\ldots, K$, define the function $q_i \in M$ which interpolates at K of the extreme points by:

$$q_i(x_k) = \operatorname{sgn}(f(x_k) - B_X(f)(x_k))$$

for $k = 0, 1,\ldots, K$ and $k \neq i$. Then,

$$\gamma(X, f) = (\max_{0 \leqslant i \leqslant K} \| q_i \|_X)^{-1} \tag{2.2}$$

The following example shows that continuity with respect to density may fail.

Example. Let $X(n)$, $n = 4, 5,\ldots$, be a sequence of subsets of $X = [0, 1]$ given by $X(n) = [1/n, 1 - 1/n]$. Let $M = \pi_1$ and let $f \in C[0, 1]$ be the piecewise linear function which satisfies $f(0) = f(1/2) = f(1) = 1$ and $f(1/4) = f(3/4) = -1$. Then we show that $\lim_{n\to\infty} d(X(n), X) = 0$ but,

$$\lim_{n\to\infty} \gamma(X(n), f) \neq \gamma(X, f).$$

First observe that $B_{X(n)}(f) = B_X(f) = 0$ and $E_n(f) = \{1/4, 1/2, 3/4\}$. Thus we can use (2.2) to calculate γ on $X(n)$ and there are three interpolating functions: $q_0(x) = -8x + 5$, $q_1(x) = -1$, and $q_2(x) = 8x - 3$, with $\| q_0 \|_{X(n)} = \| q_2 \|_{X(n)} = (5n - 8)/n$ and $\| q_1 \|_{X(n)} = 1$. Thus $\gamma(X(n), f) = n/(5n - 8)$.

To compute $\gamma(X, f)$ use (2.1). Any m in M with $\| m \|_X = 1$ satisfies $m(0) = 1$, $m(1) = 1$, $m(0) = -1$ or $m(1) = -1$. For any m in M with $m(0) = 1$ or $m(1) = 1$, the max in (2.1) will be 1. If $m(0) = -1$, then by graphing $f(x)$ and $m(x)$ one sees the max clearly occurs when $m(1/4)f(1/4) = -m(1)f(1)$ and thus $m(x) = 8/5x - 1$ and the inf in (2.1) for this m will be $3/5$. Alternatively one can obtain $3/5$ by computing

$$\inf_{0 \leqslant a \leqslant 2} \max_{x \in E_n(f)} \{f(x)\, m(x)\}$$
$$= \inf_{0 \leqslant a \leqslant 2} \max\{-1, 1 - a/4, -1 + a/2, 1 - 3a/4, a - 1\}$$

where $m(x) = ax - 1$. The case $m(1) = -1$ gives the same value as when $m(0) = -1$ by symmetry. Thus $\lim_{n \to \infty} \gamma(X(n), f) = 1/5 < \gamma(X, f) = 3/5$.

Remark. Cline (Theorem 3 in [4]) gives a computational procedure to determine some number γ to use in (1.1). This procedure involves computing for each alternating set $A_\alpha \subseteq E(f) \subseteq X$ a value,

$$\gamma(A_\alpha, f) = \inf\{\sup_{x \in A_\alpha} \operatorname{sgn}(f(x) - B_X(f)(x))\, m(x) \colon \| m \|_X = 1\}$$

utilizing the interpolation process described above (2.2). The largest possible constant arising from this procedure for which (1.1) holds would then be $\sup_\alpha \gamma(A_\alpha, f)$. Since $\gamma(A_\alpha, f) \leqslant \gamma(X, f)$ for each α, $\sup_\alpha \gamma(A_\alpha, f) \leqslant \gamma(X, f)$. The above example demonstrates that this inequality may in fact be strict. In particular, $E_n(f)$ allows five alternating sets, and $\sup_{1 \leqslant i \leqslant 5} \gamma(A_i, f) = \frac{1}{5}$.

3. Main Results

The first Theorem shows that $\gamma(X, f)$ depends continuously on X providing the extreme points $E_X(f)$ depend continuously on X. The proof is given after a Lemma and a Proposition which asserts that regardless of the behavior of $E_X(f)$, $\gamma(X, f)$ is an upper semi-continuous function of X.

THEOREM 1. *If*

$$\lim_{n \to \infty} d(X(n), X) = 0$$

and

$$\lim_{n \to \infty} d(E_n(f), E(f)) = 0,$$

then

$$\lim_{n\to\infty} \gamma(X(n), f) = \gamma(X, f).$$

PROPOSITION. *The constant γ satisfies:*

$$\lim_{d(Y,X)\to 0} \sup \gamma(Y, f) \leqslant \gamma(X, f).$$

Proof. The first part of the proof consists of showing that for any $g \in C(X)$,

$$\lim_{d(Y,X)\to 0} \| g - B_Y(f)\|_Y = \| g - B_X(f)\|_X , \tag{3.1}$$

It is well known (cf. [3]) that as $d(Y, X) \to 0$, $B_Y(f)$ converges uniformly to $B_X(f)$ on X and thus,

$$\begin{aligned}\lim_{d(Y,X)\to 0} \sup \| g - B_Y(f)\|_Y &\leqslant \lim_{d(Y,X)\to 0} \sup \| g - B_Y(f)\|_X \\ &= \lim_{d(Y,X)\to 0} \| g - B_Y(f)\|_X \\ &= \| g - B_X(f)\|_X \end{aligned} \tag{3.2}$$

Let $\epsilon > 0$ be given. We show that,

$$\lim_{d(Y,X)\to 0} \inf \| g - B_Y(f)\|_Y \geqslant \| g - B_X(f)\|_X - \epsilon.$$

Since $\{B_Y(f)\}_{Y\subseteq X}$ is equicontinuous on X for $d(Y, X)$ small enough, there exists a $\delta > 0$ such that if $| x - y | < \delta$ and $d(Y, X) < \delta$, then both the following occur:

$$| B_Y(f)(x) - B_Y(f)(y)| < \epsilon/2$$

for all $B_Y(f)$ with $d(Y, X) < \delta$ and,

$$| g(x) - g(y)| < \epsilon/2 \qquad \text{for} \quad x, y, \in X.$$

Thus for x in X,

$$\begin{aligned}| g(x)-B_Y(f)(x)| &\leqslant | g(y)-B_Y(f)(y)| + | g(x)-g(y)| + | B_Y(f)(y)-B_Y(f)(x)| \\ &\leqslant \| g - B_Y(f)\|_Y + \epsilon.\end{aligned}$$

Thus,

$$\| g - B_Y(f)\|_X \leqslant \| g - B_Y(f)\|_Y + \epsilon$$

and for any $\epsilon > 0$,

$$\|g - B_X(f)\|_X - \epsilon = \lim_{d(Y,X)\to 0} \inf \|g - B_Y(f)\|_X - \epsilon$$

$$\leqslant \lim_{d(Y,X)\to 0} \inf \|g - B_Y(f)\|_Y \tag{3.3}$$

Combining (3.2) and (3.3) yields (3.1).

Now for the second part of the proof, suppose that the conclusion is false and that therefore there are sets $\{Y(n)\}$ such that $d(Y(n), X) \to 0$ and $\lim_{n\to\infty} \sup \gamma(Y(n), f) > \gamma(X, f)$. Using an appropriate subsequence (or subnet) of $\{Y(n)\}$ assume that $\gamma(Y(n), f) \geqslant \gamma(X, f) + \epsilon$ for all $n > N$. Fix any m in M. Then by definition of strong unicity on $Y(n)$,

$$\|f - m\|_{Y(n)} \geqslant \|f - B_{Y(n)}(f)\|_{Y(n)} + \gamma(Y(n), f)\|m - B_{Y(n)}(f)\|_{Y(n)}$$

Letting $n \to \infty$ in each term and using the first part of the proof we find,

$$\|f - m\|_X \geqslant \|f - B_X(f)\|_X + (\gamma(X, f) + \epsilon)\|m - B_X(f)\|_X$$

which contradicts the fact that $\gamma(X, f)$ is the largest number for which (1.1) is valid for all $m \in M$.

LEMMA 1. *Let $X(n) \subseteq X$, $n = 1, 2,\ldots$. Assume for each $n = 1, 2,\ldots$, that $m_n \in M$ and $\|m_n\|_{X(n)} = 1$. If $\lim_{n\to\infty} d(X(n), X) = 0$, then*

$$\lim_{n\to\infty} \{\|m_n\|_X - \|m_n\|_{X(n)}\} = 0$$

Proof. There exist constants (Lemma 1, p. 85 in [3]) A and δ_1 such that if $Y \subseteq X$ and $d(Y, X) < \delta_1$, then for every m in M, $\|m\|_X \leqslant A\|m\|_Y$. Thus when $d(X(n), X) < \delta_1$ for $n > N$, we have $\|m_n\|_X \leqslant A$. Under these circumstances (Lemma 1, p. 16 [10]) there exists a constant B such that if $m_n = \sum_{i=1}^{K} a_i^{(n)}\phi_i$, where M is spanned by $\{\phi_1, \ldots, \phi_k\}$, then $|a_i^{(n)}| \leqslant B$ for $i = 1, 2,\ldots, K$ and all $n > N$. Given $\epsilon > 0$, there is a $\delta > 0$ such that

$$|\phi_i(x) - \phi_i(y)| < \epsilon/BK, \qquad i = 1, 2,\ldots, K$$

whenever $|x - y| < \delta$ and $x, y \in X$. Let $\delta < \delta_1$ and assume $d(X(n), X) < \delta$ if $n > N$. Let $\|m_n\|_X = |m_n(\bar{x}_n)|$ for some point $\bar{x}_n \in X$. Then for $n > N$,

$$\|m_n\|_X - \|m_n\|_{X(n)} \leqslant \left|\sum_{i=1}^{K} a_i^{(n)}\phi(\bar{x}_n)\right| - \left|\sum_{i=1}^{K} a_i^{(n)}\phi_i(x)\right|$$

$$\leqslant \sum_{i=1}^{K} |a_i^{(n)}|\,|\phi_i(\bar{x}_n) - \phi_i(x)| \tag{3.4}$$

for any $x \in X(n)$. For any $n > N$, there is some $x \in X(n)$ with $|\bar{x}_n - x| < \delta$. Thus (3.4) is less than ϵ.

Proof of Theorem 1. By the Proposition it suffices to show that,

$$\liminf_{n\to\infty} \gamma(X(n), f) \geqslant \gamma(X, f) \tag{3.5}$$

Let $R_n(x) = f(x) - B_{X(n)}(f)(x)$ and $R(x) = f(x) - B_X(f)(x)$. Assume without loss of generality that $\|f\|_X = 1$ and $B_X(f) = 0$. Then by (2.1) there exists for each $n = 1, 2,...,$ a function m_n in M such that $\| m_n \|_{X(n)} = 1$ and,

$$-\gamma(X(n), f) + \max_{x\in E_n(f)} \operatorname{sgn}(R_n(x))\, m_n(x) < \epsilon/4.$$

Then for any n,

$$\begin{aligned}\gamma(X, f) - \gamma(X(n), f) &\leqslant \max_{x\in E(f)} f(x)\, m_n(x)(\| m_n \|_X)^{-1} \\ &\quad - \max_{x\in E_n(f)} \operatorname{sgn}(R_n(x))\, m_n(x) \\ &\quad + \max_{x\in E_n(f)} \operatorname{sgn}(R_n(x))\, m_n(x) - \gamma(X(n), f) \\ &\leqslant \epsilon/4 + \max_{x\in E(f)} f(x)\, m_n(x)(\| m_n \|_X)^{-1} \\ &\quad - \max_{x\in E_n(f)} \operatorname{sgn}(R_n(x))\, m_n(x). \end{aligned} \tag{3.6}$$

For $n = 1, 2,...,$ let x_n' be a point where $\max_{x\in E(f)} f(x)\, m_n(x)(\| m_n \|_X)^{-1}$ is achieved. Then for any n and $x \in E_n(f)$, (3.6) implies that

$$\begin{aligned}&\gamma(X, f) - \gamma(X(n), f) \\ &\quad\leqslant \epsilon/4 + f(x_n')\, m_n(x_n')(\| m_n \|_X)^{-1} - \operatorname{sgn}(R_n(x))\, m_n(x) \\ &\quad\leqslant \epsilon/4 + | f(x_n')\, m_n(x_n')(\| m_n \|_X)^{-1} - f(x)\, m_n(x_n')(\| m_n \|_X)^{-1} | \\ &\qquad + | f(x)\, m_n(x_n')(\| m_n \|_X)^{-1} - \operatorname{sgn}(R_n(x))\, m_n(x_n')(\| m_n \|_X)^{-1} | \\ &\qquad + | \operatorname{sgn}(R_n(x))\, m_n(x_n')(\| m_n \|_X)^{-1} - \operatorname{sgn}(R_n(x))\, m_n(x)| \\ &\quad\leqslant \epsilon/4 + | f(x_n') - f(x)| + | f(x) - \operatorname{sgn}(R_n(x))| \\ &\qquad + | m_n(x_n')(\| m_n \|_X)^{-1} - m_n(x)|. \end{aligned} \tag{3.7}$$

Since $x \in E_n(f)$, it can be shown that

$$f(x) - \operatorname{sgn} R_n(x) = f(x) - (f(x) - B_{X(n)}(f)(x))(\| f - B_{X(n)}(f) \|_{X(n)})^{-1}$$

converges to zero as $n \to +\infty$ because by (3.1), $\|f - B_{X(n)}(f)\|_{X(n)}$ converges to $\|f - B_X(f)\|_X = 1$ and $B_{X(n)}(f)$ converges to $B_X(f) = 0$ uniformly on X. Thus for all n sufficiently large and any $x \in E_n(f)$, (3.7) implies that

$$\gamma(X, f) - \gamma(X(n), f)$$
$$\leqslant \epsilon/2 + |f(x_n') - f(x)| + \left[\frac{m_n(x_n')}{\| m_n \|_X} - \frac{m_n(x)}{\| m_n \|_{X(n)}}\right] \tag{3.8}$$

Recall $\| m_n \|_{X(n)} = 1$. By Lemma 1, there is an N so that if $n > N$, then $\| m_n \|_X - \| m_n \|_{X(n)} < \epsilon/8$. For any fixed $n > N$, there is an $x \in E_n(f)$ with $|f(x_n') - f(x)| < \epsilon/4$ and also $| m_n(x_n') - m_n(x)| < \epsilon/8$. Hence for $n > N$, (3.8) is less than ϵ. This shows that for any $\epsilon > 0$, there is an N such that if $n > N$, then $\gamma(X, f) - \gamma(X(n), f) < \epsilon$. Hence (3.5) follows.

THEOREM 2. *Assume that $f \notin M$, $E(f)$ has exactly* $1 + \dim M$ *points and* $\lim_{n\to\infty} d(X(n), X) = 0$. *Then,*

$$\lim_{n\to\infty} \gamma(X(n), f) = \gamma(X, f)$$

Notice that if X has at least $K + 2$ points and $E(f)$ has exactly $1 + \dim M = K + 1$ points, then $f \notin M$ follows. The proof consists of applying the following interesting Lemma to observe that $\lim_{n\to\infty} d(E_n(f), E(f)) = 0$ and then applying Theorem 1. Observe that although $E(f)$ in the Lemma has just $K + 1$ points, $E_n(f)$ might even have infinitely many.

LEMMA 2. *Assume $f \notin M$ and $E(f)$ has $K + 1$ points and for each n, let $A_n = \{x_{i,n}\}_{i=0}^K$ be some alternation set for $f - B_{X(n)}(f)$ on $X(n)$ and $A = \{x_i\}_{i=0}^K$ be the alternation set for $f - B_X(f)$. Then* $\lim_{n\to\infty} A_n = A$.

Proof. We show that $\lim_{n\to\infty} x_{in} = x_i$. We have $A_n \subseteq E_n(f) \subseteq X(n)$ and $x_{0n} < x_{1n} < \cdots < x_{Kn}$. For each $i = 0, 1,..., K$, $\{x_{in}\}_n$ contains a convergent subsequence, say $\{x_{in(j)}\}$, converging to $\bar{x}_i$, and clearly $\bar{x}_i \leqslant \bar{x}_{i+1}$. First we show $\bar{x}_i < \bar{x}_{i+1}$. Suppose to the contrary that for some i, $\bar{x}_i = \bar{x}_{i+1}$. Let $R_j = f - B_{X(n(j))}(f)$. Then $\| B_{X(n(j))}(f) - B(f)\|_X \to 0$ implies $\| R_j - R\|_X \to 0$. Let j be large enough so that,

$$| R(x_{i,n(j)}) - R(x_{i+1,n(j)})| < \| R \|_X .$$

This is possible because $x_{i,n(j)} \to \bar{x}_i = \bar{x}_{i+1}$, $x_{i+1,n(j)} \to \bar{x}_{i+1}$ and $\| R \|_X > 0$. Also select j large enough to insure that,

$$\| R - R_j \|_X < (1/8) \| R \|_X \quad \text{and} \quad \| R \|_{X(n(j))} > (3/4) \| R \|_X .$$

Then,

$$\begin{aligned}\| R \|_X &> | R(x_{i,n(j)}) - R(x_{i+1,n(j)})| \\ &\geqslant | R_j(x_{i,n(j)}) - R_j(x_{i+1,n(j)})| \\ &\quad - | R(x_{i,n(j)} - R_j(x_{i,n(j)})| - | R(x_{i+1,n(j)}) - R_j(x_{i+1,n(j)})| \\ &\geqslant 2 \| R_j \|_{X(n(j))} - 2 \| R - R_j \|_X .\end{aligned}$$

But $\| R_j \|_{X(n(j))} \geqslant \| R \|_{X(n(j))} - \| R - R_j \|_X$ and therefore,

$$\| R \|_X \geqslant 2 \| R \|_{X(n(j))} - 4 \| R - R_j \|_X .$$

Consequently,

$$\| R \|_X > 3/2 \| R \|_X - 4/8 \| R \|_X = \| R \|_X$$

a contradiction. Thus,

$$\bar{x}_0 < \bar{x}_1 < \cdots < \bar{x}_K .$$

By (3.1) we have $\lim_{j\to\infty} \| R_j \|_{X(n(j))} = \| R \|_X$ and thus,

$$\lim_{j\to\infty} | f(x_{i,n(j)}) - B_{X(n(j))}(f)(x_{i,n(j)})| = \| R \|_X$$

which implies,

$$| f(\bar{x}_i) - B_X(f)(\bar{x}_i)| = \| R \|_X \qquad i = 0,..., K.$$

Hence $\{\bar{x}_0 ,..., \bar{x}_K\} = E(f) = \{x_0 ,..., x_K\}$. Since this is true for any subsequence $\{x_{i,n(j)}\}$ it follows that the sequence $\{x_{i,n}\}_{n=1}^{\infty}$ itself converges to x_i and consequently A_n converges to A.

It is of interest to observe that in the above proof for each i,

$$\text{sgn}(f(x_{i,n(j)}) - B_{X(n(j))}(f)(x_{i,n(j)}) = \text{sgn } R(x_i)$$

for all but finitely many j. This follows because,

$$\lim_{j\to\infty} f(x_{i,n(j)}) - B_{X(n(j))}(f)(x_{i,n(j)}) = f(x_i) - B_X(f)(x_i).$$

The previous two results can be applied to the following example where in particular $E_n(f)$ is larger than $E(f)$.

EXAMPLE 2. Let $X = [0, 1]$, $X_n = [0, 1/2 - 1/n] \cup [1/2 + 1/n, 1]$, $f(x) = 4(x - 1/2)^2$ and $M = \pi_1$. Then $B_X(f)(x) = 1/2$ and $E(f) = \{0, 1/2, 1\}$ and $E(f)$ has $K + 1$ points. By the previous two results, $\lim_{n\to\infty} \gamma(X(n), f) = \gamma(X, f)$ and $\lim_{n\to\infty} d(E_n(f), E(f)) = 0$. Here $E_n(f) = \{0, 1/2 - 1/n, 1/2 + 1/n, 1\}$ and $B_{X(n)}(f)(x) = 1/2 + 2/n^2$. The alternation sets on $X(n)$ are $A_{1n} =$

$\{0, 1/2 - 1/n, 1\}$ and $A_{2n} = \{0, 1/2 + 1/n, 1\}$ and these as predicted converge to $E(f)$. Using (2.2) and (2.1) respectively one obtains $\gamma(X, f) = 1/3$ and $\gamma(X(n), f) = (n + 2)/(3n - 2)$.

COROLLARY. *Let $I_n \subseteq I$, $n = 1, 2, \ldots$, be intervals satisfying* $\lim_{n\to\infty} d(I_n, I) = 0$. *Let $M = \pi_K$, and assume $f^{(K+1)}(x) \neq 0$ on I. Then,*

$$\lim_{n\to\infty} \gamma(I_n, f) = \gamma(I, f)$$

EXAMPLES. Let $I = [-1, 1]$, $I_n = [-1 + 1/n, 1 - 1/n]$, $f(x) = e^x$ and $f_a(x) = 1/(a - x)$ for $a \geqslant 2$. Then the Corollary shows that $\gamma(I_n, f) \to \gamma(I, f)$ and $\gamma(I_n, f_a) \to \gamma(I, f_a)$.

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